Supported by tetfund TERTIARY EDUCATION TRUST FUND

On the Application of Uniformization Method for Large Spaces in Solving the Transient Distribution in Markov Chain

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Received: March 20, 2022 **Accepted:** June 18, 2022

Abstract: The computation of state probability distributions at an arbitrary point in time, which in the case of a discrete-time Markov chain means finding the distribution at some arbitrary time step n denoted$\pi^{(n)}$, a row vector whose i^{th} component is the probability that the Markov chain is in state i at time step n, is the iterative solution methods for transient distribution in Markov chain. The solutions of transient distribution in Markov chain using uniformization methods for large state spaces have been investigated in this study, in order to provide some insight into the solutions of transient distribution in Markov chain using uniformization methods for large state spaces, which produce a significantly more accurate response in less time for some types of situations and also tries to get to the end result as quickly as possible while the solution must be computed when a specified number of well-defined stages have been completed. Our goal is to compute solutions and algorithms for large state spaces using uniformization methods, which begin with an initial estimate of the solution vector and then alter it in such a way that it gets closer and closer to the true solution with each step or iteration, saving time. Matrices operations, such as multiplication with one or more vectors, are performed using Markov chain laws, theorems, and formulas.
For an illustrative example, the transient distribution vector's$\pi^{(n)}$, $n = 1, 2, \ldots,$; the value of$K$, the number of terms to be included in the summation are acquired, and the technique is well presented.
Keywords: Infinitesimal Generator, Uniformization Method, Runge-Kutta Methods, the Adams Formula, Backward Differentiation Formulae

Introduction

In the discipline of numerical analysis, there are two types of solution methods: iterative solution methods and direct solution methods. Iterative approaches start with an initial estimate of the solution vector and then alter it in such a way that it gets closer and closer to the genuine solution with each step or iteration. It eventually converges on the true solution. If there is no known initial approximation, a guess is performed or an arbitrary initial vector is used instead. The solution must be computed when a specified number of well-defined stages have been completed. For large-scale Markov chains, methods currently used to obtain transient solutions are based either on readily available differential equation solvers such as Runge-Kutta methods or the Adams formulae and backward differentiation formulae (BDF). Most of these methods experience difficulty when both $max_j|q_{jj}|$ (the largest exit rate from any state) and t (the time at which the solution is required) are large, and there appears to be little to recommend a single method for all situations. In this study we discuss the uniformization method (also called Jensen's method or the method of randomization). This method is extremely simple to program and often outperforms other methods, particularly when the solution is needed at a single time point close to the origin. If the solution is required at many points, or if plots need to be drawn to show the evolution of certain performance measures, then a method based on one of the differential equation solvers may be preferable. An extensive discussion on all these methods may be found in Stewart (1994). Romanovsky (1970) established the application and simulation of discrete Markov Chains and Moler and an Van Loan (1978) explain the nineteen dubious ways to compute the exponential of a matrix while Saff (1973) explained the degree of the best rational approximation to the exponential function and Philippe and Sidje (1993) derived the transient solution of Markov Processes by Krylov Subspaces, whereas Stewart (2009) discussed the development of Numerical Solutions of Markov chains, while Pesch *et al.*(2015) demonstrated the appropriateness of the Markov chain technique in the wind feed in Germany (2015) and Agboola (2016) demonstrated the batch processes of Markov chain in machine repair problem while Uzun and Kiral (2017) used the Markov chain model of fuzzy state to anticipate the direction of gold price movement and to estimate the probabilistic transition matrix of gold price closing returns, whereas Aziza *et al.* (2019) used the Markov chain model of fuzzy state to predict monthly rainfall data (2019). Clement (2019) demonstrated the application of Markov chain to the spread of disease infection, demonstrating that Hepatitis B became more infectious over time than tuberculosis and HIV, while Vermeer and Trilling (2020) demonstrated the application of Markov chain to journalism. Agboola (2021) introduced direct equation solving algorithms compositions of lower -upper triangular matrix and Grassmann–Taksar–Heyman for the stationary distribution of Markov chains, while Agboola, and Ayoade (2021) analysed the matrix geometric and analytical block numerical iterative methods for stationary distribution in the structured Markov chains. Agboola and Ayinde (2021) demonstrated the performance measure analysis on the states classification in Markov chain, while Agboola and Badmus (2021a) established the application of renewal reward processes in homogeneous discrete Markov chain and, Agboola and Badmus (2021b) established the application of Runge-kutta and backward differentiation methods for solving transient distribution in Markov chain. Agboola (2022a) discussed the decomposition and aggregation algorithmic numerical iterative solution methods for the stationary distribution of Markov chain, while Agboola (2022b) analysed and Applied an Irreducible Periodic Markov chain in solving random walk and gambler's ruin, Agboola and Ayinde (2022) discussed the application of successive overrelaxation algorithmic and block numerical iterative solutions for the stationary distribution in Markov chain and, Agboola and Ayoade (2022) introduced the performance measure analysis of the reachability matrix and absorption probabilities for close and open classification group of states in Markov chain, while Agboola, and Nehad (2022). Worked on the application of matrix scaling and powering methods of small state spaces for solving transient distribution in Markov chain, and Agboola and Obilade (2022) demonstrated the Relative Mix Transition Probabilities of Three Machine Each of Two Different Types in Repairman Problem with Batch

Deterministic Repairs. However, in this study, the analysis of transient distribution in Markov chain using the Uniformization Method for large spaces is considered

Notation

$\pi^{(n)}$, transient distribution; $\pi_i(t)$, the probability that the Markov chain is in state i at time t; $\pi(t)$, is the transient solution at time t; π, is the stationary distribution; $P(i)$, the transition probability matrix at step i; Q, infinitesimal generator; P, the stochastic transition probability matrix of the discrete-time Markov chain; θ, the parameter of the Poisson process; ε, a tolerance criterion; and K, the number of terms to be included in the summation

Material and Methods

The study area consisted of the analysis of transient distribution in Markov chain using the Uniformization method for large spaces. The uniformization method is a widely used and often very efficient approach that can be applied to both small dense and sparse transition rate matrices as well as large sparse transition matrices. It's not ideal for Markov chains with a lot of stiffness. In a Markov Chain, the computation of state probability distributions at any point in time is known as transient distribution. This means finding the distribution at any arbitrary time step n in the case of a discrete-time Markov chain.

The probability that the Markov chain is in state i at time step n is given by $\pi^{(n)}$, a row vector whose i^{th} component is the probability that the Markov chain is in state i at time step n. This fulfills the relationship's requirements.

$$\pi^{(n)} = \pi^{(n-1)}P(n-1) = \pi^{(0)}P(0)P(1)\cdots P(n-1) \quad (1)$$

where $P(i)$ is the step i transition probability matrix, this is reduced for a homogeneous discrete-time Markov chain to

$$\pi^{(n)} = \pi^{(n-1)}P = \pi^{(0)}P^n \quad (2)$$

where $P(0) = P(1) = \cdots = P$. We seek the distribution at any time i for a continuous-time Markov chain with infinitesimal generator *Q*. Such a distribution is denoted $\pi(t)$, a row vector whose component $\pi_i(t)$ is the probability that the Markov chain is in state i at time $t$and this vector satisfies the relationship

$$\pi(t) = \pi(0)e^{Qt} \quad (3)$$

where e^{Qt} is the matrix exponential defined by

$$e^{Qt} = \sum_{k=0}^{\infty} \frac{(Qt)^k}{k!}. \quad (4)$$

In both cases, what is usually required is the probability distribution $\pi^{(n)}$ or $\pi(t)$. The uniformization method revolves around a discrete-time Markov chain that is embedded in the continuous-time process. The transition probability matrix of this discrete-time chain is constructed as

$$P = Q\Delta t + I \quad (5)$$

With $\Delta t < \frac{\mathbf{1}}{max_i|q_{ii}|}$. In this Markov chain all state transitions occur at a uniform rate equal to $\frac{\mathbf{1}}{\Delta t}$, hence the name uniformization. Letting $\theta = max_i|q_{ii}|$, we may write

$$Q = \theta(P - I) \quad (6)$$

and inserting this into the Kolmogorov forward differential equations we get

$$\pi(t) = \pi(0)e^{Qt} = \pi(0)e^{\theta(P-I)t} = \pi(0)e^{-\theta t}e^{\theta Pt}. \quad (7)$$

Expanding $e^{\theta Pt}$ in a Taylor's series, we obtain

$$\pi(t) = \pi(0)e^{-\theta t}\sum_{k=0}^{\infty}\frac{(\theta tP)^k}{k!} = \pi(0)e^{-\theta t}\sum_{k=0}^{\infty}\frac{(\theta t)^k P^k}{k!} \quad (8)$$

$$= \pi(0)P^k\sum_{k=0}^{\infty}\frac{(\theta t)^k}{k!}e^{-\theta t}. \quad (9)$$

There are two points to consider. First, the term $\pi(0)P^k$may be regarded as the vector that provides the probability distribution after *k* steps of a discrete-time Markov chain with stochastic transition probability matrix P and initial distribution $\pi(0)$. Second, the Poisson process with rate θ defined as the probability of k events happening in [0, *t*) yields the expression $e^{-\theta t}\frac{(\theta t)^k}{k!}$ Which is the probability that the discrete-time Markov chain will take k transition steps in the interval [0, *t*). These probabilities can be interpreted as weights that, when multiplied by the discrete-time Markov chain's distribution after k steps and summed over all possible number of steps, yield the transient distribution $\pi(t)$. The uniformization method calculates the distribution $\pi(t)$ directly from Equation (9). Writing it in the form

$$\pi(t) = e^{-\theta t}\sum_{k=0}^{\infty}\left(\pi(0)P^{k-1}\frac{(\theta t)^{k-1}}{(k-1)!}\right)P\frac{\theta t}{k} \quad (10)$$

exposes a convenient recursive formulation. Setting $x = \pi = \pi(0)$ \ and iterating sufficiently long with

$$x = x\left(P\frac{\theta t}{k}\right), \quad \pi = \pi + x \quad (11)$$

allows $\pi(t) = e^{-\theta t}\pi$ to be used to construct the transient distribution. $\pi(t)$ might be derived more readily straight from the Chapman-Kolmogorov differential equations using the formula.

$$\pi(t) = \pi(0)\sum_{k=0}^{\infty}\frac{(Qt)^k}{k!}. \quad (12)$$

However, because the matrix Q contains both positive and negative components, some of which may be bigger than one, the algorithm is less stable than one based on the matrix P, which, as a stochastic matrix, has all positive members in the range [0, 1]. Hence, Equation (9) is appropriate. The uniformization technique's numerical advantages include its simplicity of translation into computer code and the control it provides over the truncation error. We need to truncate the infinite series in order to fix the truncation mistake in the uniformization equation (9).

$$\pi^*(t) = \sum_{k=0}^{K}\pi(0)P^k\frac{(\theta t)^k}{k!}e^{-\theta t}. \quad (13)$$

Also, let $\tau(t) = \pi(t) - \pi^*(t)$. $||\tau(t)||$ is the truncation error for any consistent vector norm. This inaccuracy can easily be quantified mathematically. If we choose *K* to be large enough,

$$1 - \sum_{k=0}^{K}\frac{(\theta t)^k}{k!}e^{-\theta t} \le \varepsilon$$

or, equivalently, that

$$\sum_{k=0}^{K}\frac{(\theta t)^k}{k!} \ge \frac{1-\varepsilon}{e^{-\theta t}} = (1-\varepsilon)e^{\theta t}, \quad (14)$$

Where ε is a predetermined truncation condition, it follows that

$$\|\pi(t) - \pi^*(t)\|_\infty \le \varepsilon \quad (15)$$

To see this, observe that

$$\|\pi(t) - \pi^*(t)\|_\infty = \left\|\sum_{k=0}^{\infty}\pi(0)P^k\frac{(\theta t)^k}{k!}e^{-\theta t} - \sum_{k=0}^{K}\pi(0)P^k\frac{(\theta t)^k}{k!}e^{-\theta t}\right\|_\infty$$

$$= \left\|\sum_{k=K+1}^{\infty}\pi(0)P^k\frac{(\theta t)^k}{k!}e^{-\theta t}\right\|_\infty \le \sum_{k=0}^{K}\frac{(\theta t)^k}{k!}e^{-\theta t}$$

$$= \sum_{k=0}^{\infty}\frac{(\theta t)^k}{k!}e^{-\theta t} - \sum_{k=0}^{K}\frac{(\theta t)^k}{k!}e^{-\theta t}$$

$$= 1 - \sum_{k=0}^{K}\frac{(\theta t)^k}{k!}e^{-\theta t} \le \varepsilon. \quad (16)$$

Results and Discussion

This section discusses the derivation of formulae for performance measures obtained from the illustrative example such as stationary distribution π, transient distribution $\pi^{(n)}$,

Illustrative Example: Consider a Markov chain with an infinitesimal generator that runs indefinitely.

$$Q = \begin{pmatrix} -5 & 2 & 3 \\ 1 & -2 & 1 \\ 6 & 4 & -10 \end{pmatrix}$$

With $\pi(0) = (1 \quad 0 \quad 0)$ and $t = 1$, we want to see how Equation (1) behaves as the number of terms in the summation grows. The uniformization method yields a value of $\theta = 10$ and

$$P = \begin{pmatrix} 0.5 & 0.2 & 0.3 \\ 0.1 & 0.8 & 0.1 \\ 0.6 & 0.44 & 0 \end{pmatrix}$$

Assuming we need a precision of tenths of a $\varepsilon = 10^{-6}$. To determine the value of K, the number of terms to be included in the summation, we proceed step by step, increasing k until the desired result is obtained.

$$\vartheta_K = \sum_{k=0}^{K} \frac{(\theta t)^k}{k!} \geq (1-\varepsilon)e^{\theta t} = (1-10^{-6})e^{10} = 22{,}026.44.$$

Observe that successive terms in the summation satisfy

$$\omega_{K+1} = \omega_K \frac{\theta t}{K+1} \quad with \quad \omega_0 = 1 \qquad (17)$$

and that

$$\vartheta_{K+1} = \vartheta_K + \omega_{K+1} \; with \; \vartheta_0 = 1, \qquad (18)$$

and so, beginning with K = 0, and using this recursion, we successively compute

$\vartheta_0 = 1, \quad \vartheta_1 = 11, \quad \vartheta_2 = 61, \quad \vartheta_3 = 227.667, \quad \vartheta_4 = 644.333, \; \cdots, \; \vartheta_{28} = 22{,}026.45,$

As a result, the summing requires K = 28 terms. We must compute to get our estimate to the transient distribution at time t = 1

$$\pi(t) \approx \pi(0)P^k \textstyle\sum_{k=0}^{28} \frac{(\theta t)^k}{k!} e^{-\theta t}. \qquad (19)$$

for some initial distribution, $\pi(0) = (1 \quad 0 \quad 0)$. Using the recursion relation of Equation (11), we find

$$k = 0 : x = (1,0,0); \; \pi = (1,0,0),$$
$$k = 1 : x = (5,2,3); \; \pi = (6,2,3),$$
$$k = 2 : x = (22.5, 19, 8.5); \; \pi = (28.5, 21, 11.5),$$
$$k = 3 : x = (60.8333, 77, 28.8333); \; \pi = (89.3333, 98, 40.3333),$$
$$\vdots$$
$$k = 28 : x = (.009371, .018742, .004686); \; \pi = (6, 416.9883, 12, 424.44968, 3, 184.4922).$$

In this example, the elements of x increase until they reach

$$x = (793.8925, 1, 566.1563, 395.6831)$$

For $k = 10$ and then they begin to decrease. Multiplying the final π vector by e^{-10} produces the desired transient distribution:

$$\pi(1) = e^{-10}(6, 416.9883, 12, 424.44968, 3, 184.4922) = (.291331, .564093, .144576).$$

We can code Equation (9) exactly as it appears in the uniformization technique, with the understanding that $\pi(0)P_k$ is computed iteratively, i.e., we do not construct the k^{th} power of P and pre-multiply it with $\pi(0)$, but instead form the sequence of vectors, $\psi(j+1) = \psi(j)P$, with $\psi(0) = \pi(0)$, so that $\pi(0)P_k$ is given by $\psi(k)$. We could also divide Equation (9) into time steps $0 = t_0, t_1, t_2, t_3, \cdots, t_m = t$ and create code to implement them.

$$\pi(t_{i+1}) = \textstyle\sum_{k=K+1}^{\infty} \pi(t_i)P^k \frac{(\theta)^k (t_{i+1}-t_i)^k}{k!} e^{-\theta(t_{i+1}-t_i)} \qquad (20)$$

recursively for $i = 0, 1, \ldots, (m-1)$. If the transient solution is required at various points $t_1, t_2, \cdots$, between the initial time t_0 and the end time t, the second approach is the obvious way to execute the computation. If the transient solution is only required at a single terminal point, it is computationally more expensive. When the numerical values of θ and t are such that the computer underflows when computing $e^{-\theta(t)}$, it may be useful.

Such occurrences can be foreseen in advance and appropriate action taken. One might, for example, elect not to allow t values to exceed 100. When this happens, divide the time t into $d = 1 + \left(\frac{\theta t}{100}\right)$ equal intervals and compute the transient solution at times$t/d, 2t/d, 3t/d, \ldots, t$.. When implementing such a procedure, caution must be exercised because errors in the computation of intermediate values $\pi(t_i)$ may propagate to subsequent values$\pi(t_j)$, $j > i$. An alternative to dividing a large interval θt into more manageable pieces may be to omit from the summation in Equation (16) expressions for which the value of $e^{-\theta t}\frac{(\theta t)^k}{k!}$ is so small that it may generate numerical difficulties. This can be performed by selecting a left truncation point, l, as the largest value for which

$$\textstyle\sum_{k=0}^{l-1} \frac{(\theta t)^k}{k!} e^{-\theta t} \leq \varepsilon_i \qquad (21)$$

The needed transient distribution vector is now computed as for some lower limitε_i.

$$\pi^*(t) = \textstyle\sum_{k=0}^{K} \psi(k) \frac{(\theta t)^k}{k!} e^{-\theta t} \qquad (22)$$

as previously, with $\psi(k)$ computed recursively, the value of l can be easily calculated using the same approach as the upper limit of the summation, K. i.e., from $k = 0$ to $k = K$ must be computed in all circumstances, the amount of effort involved in summing from $k = l$ is essentially the same as summing from$k = 0$. Since $\psi(k)$ from $k = 0$ to $k = K$ must be computed in all cases. The only reason in summing from $k = l$ is that of computational stability. One other wrinkle may be added to the uniformization method when it is used to compute transient distributions at large values of θt —the stationary distribution of the uniformized chain may be reached well before the last term in the equation (13). If this is the case, it means that, from the point at which steady state is reached, the valu**es** $\psi(k)$ no longer change. It is possible to monitor convergence of the uniformized chain and to determine the point at which it reaches steady state. Assume this happens when k is set to k_s. After that, the transient distribution at time t can be estimated more efficiently as

$$\pi^*(t) = \textstyle\sum_{k=1}^{k_s} \psi(k) \frac{(\theta t)^k}{k!} e^{-\theta t} + \left(\sum_{k=k_s+1}^{K} \frac{(\theta t)^k}{k!} e^{-\theta t}\right) \psi(k_s) \qquad (23)$$

The following two-step algorithm computes the transient solution $\pi(t)$ at time t given the probability distribution $\pi(0)$ at time $t = 0$; P, the discrete-time Markov chain's stochastic transition probability matrix; θ, the Poisson process's parameter; and, ε, a tolerance condition. It is intended for use when the Markov chain has a large number of states. The matrix P is only used in one operation: multiplication with a vector. This implementation does not include a lower limit or a test for steady-state convergence. The algorithm can be computed as follows:

1. Use Equation (18) to compute K, the number of terms in the summation:

• Set $K = 0$; $\omega = 1$; $\vartheta = 1$; $\eta = (1-\varepsilon)/e^{-\theta t}$.

• While $\vartheta < \eta$ do

◦ Compute $K = K + 1$; $\omega = \omega \times (\theta t)/K$; $\vartheta = \vartheta + \xi$.

2. Approximate π(t) from Equation (13)*:*

• Set $\pi = \pi(0)$; $x = \pi(0)$.

• For $k = 1 \; to \; K$ do

◦ Compute $x = xP \times (\theta t)/k$; $\pi = \pi + x$.

• Compute $\pi(t) = e^{-\theta t}\pi$.

Conclusion

In this study, The solutions of transient distribution in Markov chain using uniformization methods for large state spaces, which produce a significantly more accurate response in less time for some types of situations and also, tries to get to the end result as quickly as possible while the solution must be computed when a specified number of well-defined stages have been completed has been investigated, in order to provide some insight into the solutions of transient distribution in Markov chain. Our goal is to compute the solutions and algorithms using uniformization methods for large state spaces that start with an initial estimate of the solution vector and then alter it in such a way that it gets closer and closer to the genuine solution with each step or iteration, which saves time. Matrices operations such as multiplication with one or more vectors, are used with the help of some existing laws, theorems and formulas of Markov chain. The transient distribution vector's $\pi^{(n)}$, $n = 1, 2, ...,$; the value of K, the number of terms to be included in the summation are obtained for an illustrative example and the algorithm is well presented.

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